Amherst College
Department of Mathematics and Statistics

MATH 105 | MIDTERM 1 | FALL 2018

NAME: Solutions

Read This First!

- Keep cell phones off and out of sight.
- Do not talk during the exam.
- You are allowed one page of notes, front and back. No other books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.
- Please read each question carefully. Show **ALL** work clearly in the space provided. There is an extra page at the back for additional scratchwork.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

Grading - For Instructor Use Only

Question:	1	2	3	4	5	6	Total
Points:	18	16	5	14	7	0	60
Score:							

1. [18 points] Evaluate the following limits. Answer either as a specific value, $+\infty$, $-\infty$, or "DNE."

(a) $\displaystyle\lim_{x\to 4} \frac{x^2-x-12}{x^2-3x-4}$ $\displaystyle = \lim_{x\to 4} \frac{\cancel{(x-4)}(x+3)}{\cancel{(x-4)}(x+1)}$

$\displaystyle = \frac{4+3}{4+1} = \boxed{7/5}$

(b) $\displaystyle\lim_{\underbrace{x\to 3^-}_{x<3}} \frac{x-5}{x-3}$ $\displaystyle = \frac{3-5}{0^-} = \frac{-2}{0^-} = \boxed{+\infty}$

means

$x-3<0$

(c) $\displaystyle\lim_{x\to -1} \frac{x^2-1}{\sqrt{x+5}-2} \cdot \frac{\sqrt{x+5}+2}{\sqrt{x+5}+2} = \lim_{x\to -1} \frac{(x+1)(x-1)(\sqrt{x+5}+2)}{(x+5)-4}$

$\displaystyle = \lim_{x\to -1} \frac{\cancel{(x+1)}(x-1)(\sqrt{x+5}+2)}{\cancel{x+1}} = (-2)\cdot(\sqrt{4}+2)$

$= \boxed{-8}$

(d) $\lim_{x\to 2} \frac{x^2+2x-7}{x^2-9}$ $= \frac{4+4-7}{4-9} = \frac{1}{-5}$

$= \boxed{-1/5}$

(e) $\lim_{x\to 1} \frac{x^2-3x+2}{|x-1|}$ DSP gives 0/0

Consider the one-sided limits in order to simplify further:

$x<1 \Rightarrow |x-1| = -(x-1)$ $\lim_{x\to 1^-} \frac{x^2-3x+2}{|x-1|} = \lim_{x\to 1^-} \frac{\cancel{(x-1)}(x-2)}{-\cancel{(x-1)}} = \frac{-1}{-1} = 1$

$x>1 \Rightarrow |x-1| = (x-1)$ $\lim_{x\to 1^+} \frac{x^2-3x+2}{|x-1|} = \lim_{x\to 1^+} \frac{\cancel{(x-1)}(x-2)}{\cancel{(x-1)}} = \frac{-1}{1} = -1$

These disagree, so the overall limit $\boxed{\text{DNE}}$

(f) $\lim_{x\to 1^+} \frac{x^2+4x+3}{x^2-4x+3}$ DSP gives $\frac{8}{0}$; do sign analysis.

$= \lim_{x\to 1^+} \frac{(x+1)(x+3)}{(x-1)(x-3)} = \frac{2\cdot 4}{0^+\cdot(-2)} = \boxed{-\infty}$

2. [16 points] Consider the following piecewise function.

$$f(x) = \begin{cases} 4 & x \le -2 \\ 4 - x^2 & -2 < x \le 0 \\ 1 + \frac{1}{x} & 0 < x < 1 \\ 3 - x & x \ge 1 \end{cases}$$

(a) Determine all discontinuities of this function.

The individual pieces are all continuous, except $1+\frac{1}{x}$, which is discontinuous at $x=0$, but this is outside the interval where it is used.

So we need only check the break-points: $x=-2, 0, \& 1$.

$\underline{x=-2}$

LHL: $\lim_{x\to(-2)^-} f(x) = \lim_{x\to(-2)^-} (4) = 4$

RHL: $\lim_{x\to(-2)^+} f(x) = \lim_{x\to(-2)^+} (4-x^2) = 4-(-2)^2 = 0$

These differ, so there's a jump discontinuity at $x=-2$.

$\underline{x=0}$

LHL: $\lim_{x\to0^-} f(x) = \lim_{x\to0^-} (4-x^2) = 4-0^2=4$

RHL: $\lim_{x\to0^+} f(x) = \lim_{x\to0^+} (1+\frac{1}{x}) = 1+\frac{1}{0^+} = +\infty$

One of these is infinite, so there's an infinite discontinuity at $x=0$.

$\underline{x=1}$

LHL: $\lim_{x\to1^-} f(x) = \lim_{x\to1^-} (1+\frac{1}{x}) = 1+\frac{1}{1} = 2$

RHL: $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (3-x) = 3-1 = 2$

these agree & equal $f(1)=3-1=2$, so $f(x)$ is continuous here.

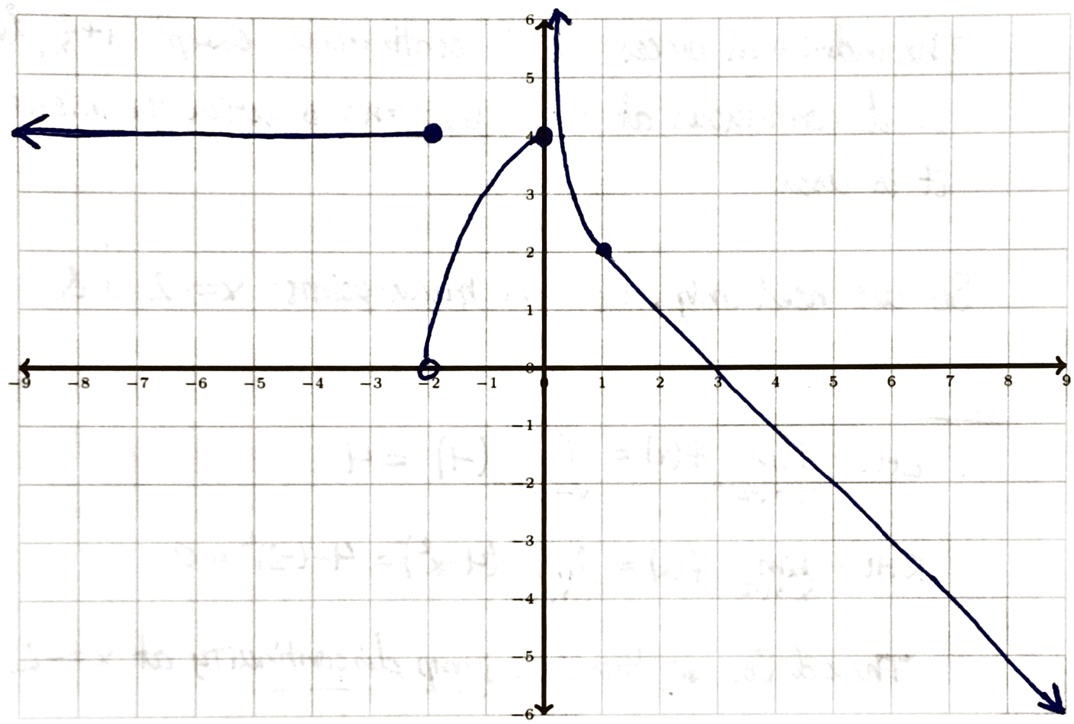
(continued on reverse)

Discontinuities at $\boxed{x=-2 \text{ and } x=0}$.

The definition of $f(x)$ is reproduced below for convenience.

$$f(x) = \begin{cases} 4 & x \leq -2 \\ 4 - x^2 & -2 < x \leq 0 \\ 1 + \frac{1}{x} & 0 < x < 1 \\ 3 - x & x \geq 1 \end{cases}$$

(b) Sketch the graph of this function on the axes below.



3. [5 points] Find an equation for a line that is parallel to the line $3x+4y = 24$ and passes through the point $(1, 2)$.

$$3x+4y=24$$

$$\Leftrightarrow \; 4y=-3x+24$$

$$\Leftrightarrow \; y=-\frac{3}{4}x+6$$

which has slope $-3/4$.

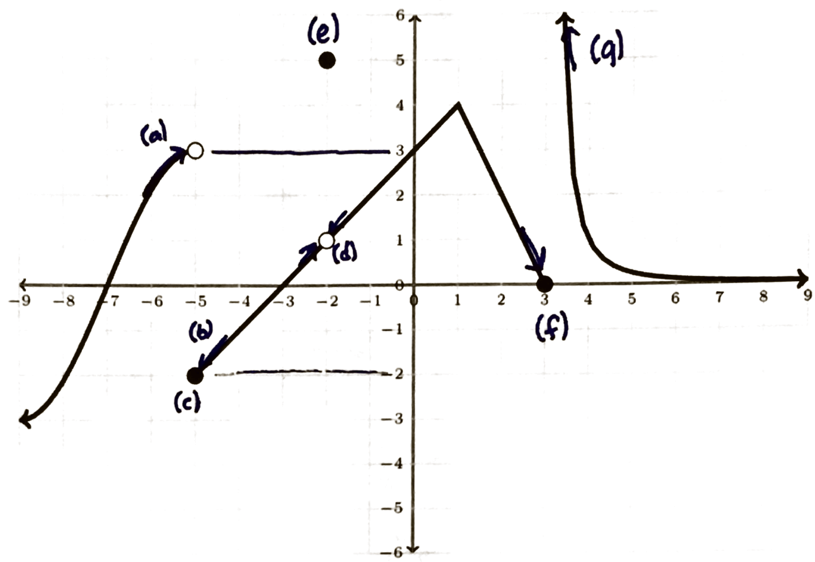
Parallel lines all have slope $-3/4$. So the desired line, in point-slope form, is

$$(y-2)=-\frac{3}{4}(x-1)$$

ie. $y-2=-\frac{3}{4}x+\frac{3}{4}$

$$\Leftrightarrow \; \boxed{y=-\frac{3}{4}x+\frac{11}{4}}$$

4. [14 points] Shown below is the graph of a function $f(x)$. Determine the following limits of function values. If a value is $+\infty$ or $-\infty$, state this. If a value does not exist, answer "DNE."



(a) $\lim\limits_{x\to(-5)^-} f(x)$

$\boxed{3}$

(b) $\lim\limits_{x\to(-5)^+} f(x)$

$\boxed{-2}$

(c) $f(-5)$

$\boxed{-2}$

(d) $\lim\limits_{x\to(-2)} f(x)$

$\boxed{1}$

(e) $f(-2)$

$\boxed{5}$

(f) $\lim\limits_{x\to 3^-} f(x)$

$\boxed{0}$

(g) $\lim\limits_{x\to 3^+} f(x)$

$\boxed{+\infty}$

(h) Identify all values of x where $f(x)$ is discontinuous.

$x=-5$ (jump)
$x=-2$ (removeable)
$x=3$ (infinite)

5. [7 points] Define $f(x)$ and $g(x)$ as follows.

$$f(x) = \frac{2+x}{1-x}$$

$$g(x) = \frac{3x+1}{x-2}$$

Compute and simplify $f(g(x))$ as much as possible.

$$f(g(x)) = \frac{2+g(x)}{1-g(x)}$$

$$= \frac{2+\frac{3x+1}{x-2}}{1-\frac{3x+1}{x-2}}$$

$$= \frac{\frac{2(x-2)+(3x+1)}{x-2}}{\frac{(x-2)-(3x+1)}{x-2}}$$

$$= \frac{2x-4+3x+1}{\cancel{x-2}} \cdot \frac{\cancel{x-2}}{x-2-3x-1}$$

$$= \frac{5x-3}{-2x-3}$$

$$= \boxed{-\frac{5x-3}{2x+3}}$$

6. [3 points (bonus)] Let $f(x) = 2 - \sqrt{3x+1}$. Compute

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Note that the answer will be in terms of x.

$$\lim_{h\to 0} \frac{\left(2-\sqrt{3(x+h)+1}\right) - \left(2-\sqrt{3x+1}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2} - \sqrt{3x+3h+1} - \cancel{2} + \sqrt{3x+1}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3x+1} - \sqrt{3x+3h+1}}{h} \cdot \frac{\sqrt{3x+1} + \sqrt{3x+3h+1}}{\sqrt{3x+1} + \sqrt{3x+3h+1}}$$

$$= \lim_{h\to 0} \frac{(3x+1) - (3x+3h+1)}{h\cdot\left(\sqrt{3x+1} + \sqrt{3x+3h+1}\right)}$$

$$= \lim_{h\to 0} \frac{-3\cancel{h}}{\cancel{h}\left(\sqrt{3x+1} + \sqrt{3x+3h+1}\right)}$$

$$= \frac{-3}{\left(\sqrt{3x+1} + \sqrt{3x+1}\right)}$$

$$= \boxed{-\frac{3}{2\sqrt{3x+1}}}$$